

NONCOHERENT INITIAL IDEALS IN EXTERIOR ALGEBRAS

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ABSTRACT. We construct a noncoherent initial ideal of an ideal in the exterior algebra of order 6, answering a question of D. Maclagan (2000). We also give a method for constructing noncoherent initial ideals in exterior algebras using certain noncoherent term orders.

1. INTRODUCTION

Let \mathbb{F} be an algebraically closed field of characteristic $\neq 2$. The **exterior algebra** \bigwedge of order n over \mathbb{F} consists of polynomials with coefficients from \mathbb{F} in noncommuting indeterminates x_1, \dots, x_n subject to the relation $x_i x_j = -x_j x_i$ for $1 \leq i, j \leq n$. This relation implies $x_i^2 = 0$ and that any monomial can be reordered up to a sign change into a canonical form $x_{i_1} \dots x_{i_k}$ where $i_1 < \dots < i_k$. Throughout, let $\mathbf{x}^{\mathbf{a}}$ denote $x_1^{a^{(1)}} \dots x_n^{a^{(n)}}$, where $\mathbf{a} = (a^{(1)}, \dots, a^{(n)}) \in \mathbb{N}^n$. Then $M_{\bigwedge} = \{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \{0, 1\}^n\}$ is the set of monomials of \bigwedge .

A **term order** \prec on M_{\bigwedge} is a total order on M_{\bigwedge} which satisfies:

- (i) $1 = \mathbf{x}^0 \prec \mathbf{x}^{\mathbf{a}}$ for all $\mathbf{x}^{\mathbf{a}} \neq 1$ in M_{\bigwedge} .
- (ii) If $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$ then $\mathbf{x}^{\mathbf{a}+\mathbf{c}} \prec \mathbf{x}^{\mathbf{b}+\mathbf{c}}$ whenever $\mathbf{x}^{\mathbf{a}+\mathbf{c}}$ and $\mathbf{x}^{\mathbf{b}+\mathbf{c}}$ are both in M_{\bigwedge} .

A term order \prec on M_{\bigwedge} is **coherent** if there exists a weight vector $w \in \mathbb{R}^n$ such that $w \cdot \mathbf{a} < w \cdot \mathbf{b}$ whenever $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$, and **noncoherent** otherwise. Equivalently, a term order on M_{\bigwedge} is coherent if it can be extended to a term order on the monomials of the usual (commutative) polynomial algebra $\mathbb{F}[x_1, \dots, x_n]$. When $n \geq 5$, there exist noncoherent term orders on M_{\bigwedge} . In the interpretation of term orders on M_{\bigwedge} as comparative probability orders on subsets of an n -element set, this fact has long been known ([KraPraSei59]).

Let $(\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_N}) =_0 (\mathbf{x}^{\mathbf{b}_1}, \dots, \mathbf{x}^{\mathbf{b}_N})$ mean that $\mathbf{x}^{\mathbf{a}_i}, \mathbf{x}^{\mathbf{b}_i} \in M_{\bigwedge}$ for all $1 \leq i \leq N$ and $\sum_{1 \leq i \leq N} \mathbf{a}_i = \sum_{1 \leq i \leq N} \mathbf{b}_i$. The following condition, when added to the defining conditions (i), (ii) of a term order on M_{\bigwedge} , results in a set of conditions which are both necessary and sufficient for coherency:

- (iii) For all $N \geq 2$ and all $\mathbf{x}^{\mathbf{a}_i}, \mathbf{x}^{\mathbf{b}_i} \in M_{\bigwedge}$, if $(\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_N}) =_0 (\mathbf{x}^{\mathbf{b}_1}, \dots, \mathbf{x}^{\mathbf{b}_N})$ and $\mathbf{x}^{\mathbf{a}_i} \prec \mathbf{x}^{\mathbf{b}_i}$ for all $i < N$, then it is not the case that $\mathbf{x}^{\mathbf{a}_N} \prec \mathbf{x}^{\mathbf{b}_N}$.

The equivalent formulation of this condition in the setting of comparative probability orders can be found in, e.g., [Fis96]. Violation of (iii) for a certain N is called a failure of the N^{th} **cancellation condition**, denoted C_N , and implies the order is noncoherent.

Let \prec be a term order and let $f = \sum_i c_i \mathbf{x}^{\mathbf{a}_i} \in \bigwedge$, where $0 \neq c_i \in \mathbb{F}$ and $\mathbf{x}^{\mathbf{a}_i} \in M_{\bigwedge}$. Then the **initial monomial** $\text{in}_{\prec}(f)$ of f is $\max_i \mathbf{x}^{\mathbf{a}_i}$ (where the maximum is taken with respect to \prec), and the **lead term** $LT_{\prec}(f)$ of f is $\max_i c_i \mathbf{x}^{\mathbf{a}_i}$. Let $I \subseteq \bigwedge$ be a left ideal. Then the **initial ideal** $\text{in}_{\prec}(I)$ of I with respect to \prec is the monomial ideal left-generated by $\{\text{in}_{\prec}(f) : f \in I\}$. An initial ideal $\text{in}_{\prec}(I)$ with respect to some noncoherent term order \prec

is a **noncoherent initial ideal** if $\text{in}_{\prec}(I) \neq \text{in}_{\prec_c}(I)$ for any coherent term order \prec_c . We will work only with homogeneous ideals $I \subset \bigwedge$. These are in fact two-sided, so we may drop the word “left” from the discussion (see, e.g., [Sto90, Section 7]).

In [Mac00], D. Maclagan posed the following question:

Does there exist a *noncoherent initial ideal* of an ideal I in the exterior algebra? That is, is there some initial ideal of I with respect to some noncoherent term order which is not equal to the initial ideal of I with respect to any coherent term order?

We give an affirmative answer to this question. For $\mathbf{x}^{\mathbf{a}} \in M_{\bigwedge}$, let $|\mathbf{a}|$ denote the sum of the entries of \mathbf{a} . Let \mathcal{P} denote the lattice of elements of $\{0, 1\}^n$, in which $\mathbf{a} < \mathbf{b}$ whenever $\mathbf{b} - \mathbf{a} \in \{0, 1\}^n$ (this is the same as ordering the subsets of an n -element set by inclusion). By an **antichain** in this lattice, we mean a set of pairwise incomparable elements.

Theorem 1.1. *Suppose \prec is a noncoherent term order with a C_N failure: $(\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_N}) =_0 (\mathbf{x}^{\mathbf{b}_1}, \dots, \mathbf{x}^{\mathbf{b}_N})$ and $\mathbf{x}^{\mathbf{a}_i} \prec \mathbf{x}^{\mathbf{b}_i}$ for all $1 \leq i \leq N$. If $|\mathbf{a}_i| = |\mathbf{b}_i|$ for all $1 \leq i \leq N$, and $\{\mathbf{a}_1, \dots, \mathbf{a}_N, \mathbf{b}_1, \dots, \mathbf{b}_N\}$ is an antichain in \mathcal{P} , then the initial ideal $\text{in}_{\prec}(I)$ of the (homogeneous) ideal*

$$I = \langle \mathbf{x}^{\mathbf{b}_i} - \mathbf{x}^{\mathbf{a}_i}, 1 \leq i \leq N \rangle \subset \bigwedge$$

is a noncoherent initial ideal.

It is not obvious that there exists an order \prec satisfying the hypotheses of Theorem 1.1. We exhibit an example of such an order, which was found using the MAGMA computer algebra system [BosCanPla97].

Example 1.2. Let \prec denote the following term order on the exterior algebra of order 6:

$$\begin{aligned} 1 &\prec x_1 \prec x_2 \prec x_3 \prec x_1x_2 \prec x_1x_3 \prec x_4 \prec x_5 \prec x_1x_4 \prec x_6 \prec x_2x_3 \prec x_1x_5 \prec \\ &x_1x_6 \prec x_1x_2x_3 \prec x_2x_4 \prec x_2x_5 \prec x_3x_4 \prec x_1x_2x_4 \prec x_3x_5 \prec x_2x_6 \prec \boxed{x_1x_2x_5 \prec x_1x_3x_4} \prec \\ &x_3x_6 \prec \boxed{x_1x_3x_5 \prec x_1x_2x_6} \prec x_4x_5 \prec x_1x_3x_6 \prec x_4x_6 \prec \boxed{x_2x_3x_4 \prec x_1x_4x_5} \prec x_5x_6 \prec \\ &\boxed{x_1x_4x_6 \prec x_2x_3x_5} \prec \dots \end{aligned}$$

(the remaining comparisons are determined since $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}} \iff \mathbf{x}^{1-\mathbf{b}} \prec \mathbf{x}^{1-\mathbf{a}}$ when $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in M_{\bigwedge}$).

The four boxed comparisons

$$\mathbf{x}^{\mathbf{a}_1} = x_1x_2x_5 \prec x_1x_3x_4 = \mathbf{x}^{\mathbf{b}_1}, \quad \mathbf{x}^{\mathbf{a}_2} = x_1x_3x_5 \prec x_1x_2x_6 = \mathbf{x}^{\mathbf{b}_2},$$

$$\mathbf{x}^{\mathbf{a}_3} = x_2x_3x_4 \prec x_1x_4x_5 = \mathbf{x}^{\mathbf{b}_3}, \quad \mathbf{x}^{\mathbf{a}_4} = x_1x_4x_6 \prec x_2x_3x_5 = \mathbf{x}^{\mathbf{b}_4}$$

are a failure of C_4 satisfying the hypotheses of Theorem 1.1. Let $I = \langle \mathbf{x}^{\mathbf{b}_i} - \mathbf{x}^{\mathbf{a}_i} : 1 \leq i \leq 4 \rangle$. Using the Gröbner basis algorithm of [Sto90, Theorem 6.6], we compute

$$\text{in}_{\prec}(I) = \langle \mathbf{x}^{\mathbf{b}_1}, \mathbf{x}^{\mathbf{b}_2}, \mathbf{x}^{\mathbf{b}_3}, \mathbf{x}^{\mathbf{b}_4}, x_1x_3x_5x_6, x_2x_3x_4x_6 \rangle.$$

It is actually easy to see that $\text{in}_{\prec}(I)$ is noncoherent. By condition (iii), at least one of $\mathbf{x}^{\mathbf{a}_1}, \mathbf{x}^{\mathbf{a}_2}, \mathbf{x}^{\mathbf{a}_3}, \mathbf{x}^{\mathbf{a}_4}$ must appear in $\text{in}_{\prec_c}(I)$ for any coherent term order \prec_c . It is clear that none of these monomials are in $\text{in}_{\prec}(I)$.

2. PROOF OF THEOREM 1.1

For $\mathbf{x}^a, \mathbf{x}^b \in M_\wedge$, we say \mathbf{x}^a **divides** \mathbf{x}^b if $\mathbf{b} - \mathbf{a} \in \{0, 1\}^n$, and by $\frac{\mathbf{x}^b}{\mathbf{x}^a}$ we mean the monomial $\mathbf{x}^{\mathbf{b}-\mathbf{a}}$. By the **least common multiple** of $c_i \mathbf{x}^a$ and $c_j \mathbf{x}^b$, we mean $\text{lcm}(c_i, c_j) \cdot \mathbf{x}^{a \cup b}$, where $(a \cup b)^{(i)} = 0$ if $a^{(i)} = b^{(i)} = 0$, and $(a \cup b)^{(i)} = 1$ otherwise.

Fix a term order \prec and let $G = \{g_1, \dots, g_r\} \subset \wedge$. Suppose $\mathbf{x}^c \text{in}_{\prec}(g_i) = 0$ for some $g_i \in G$, $\mathbf{x}^c \in M_\wedge$. Following the notation of [Mac00], define the **T -polynomial** T_{g_i, \mathbf{x}^c} to be the polynomial

$$T_{g_i, \mathbf{x}^c} = \mathbf{x}^c g_i.$$

Let m_{g_i, g_j} be the least common multiple of $LT_{\prec}(g_i)$ and $LT_{\prec}(g_j)$. Then the **S -polynomial** S_{g_i, g_j} is the polynomial

$$S_{g_i, g_j} = (-1)^{d_1} \frac{m_{g_i, g_j}}{LT_{\prec}(g_i)} g_i - (-1)^{d_2} \frac{m_{g_i, g_j}}{LT_{\prec}(g_j)} g_j.$$

where $d_1 = 1$ if reordering $\frac{m_{g_i, g_j}}{LT_{\prec}(g_i)} \text{in}_{\prec}(g_i)$ into canonical form changes the sign of this monomial, $d_1 = 0$ otherwise, and d_2 is defined similarly.

Let $f \in \wedge$. By **reducing** f with respect to G and \prec , we mean choosing some $g_i \in G$ such that $\text{in}_{\prec}(g_i)$ divides $\text{in}_{\prec}(f)$, letting $r = f - (-1)^d \frac{LT_{\prec}(f)}{LT_{\prec}(g_i)} g_i$ (where $d = 1$ if reordering $\frac{\text{in}_{\prec}(f)}{\text{in}_{\prec}(g_i)} \text{in}_{\prec}(g_i)$ changes the sign of this monomial, $d = 0$ otherwise), and repeating this process with the new polynomial r until we obtain either zero or a polynomial whose initial monomial is not divisible by any $\text{in}_{\prec}(g_j)$. Call the resulting polynomial a **remainder** of f with respect to G and \prec . This process is the same as the reduction algorithm found in [Sto90, Section 4]. It also agrees (up to the sign $(-1)^d$) with the usual reduction algorithm for elements of $\mathbb{F}[x_1, \dots, x_n]$ found in, for example, [CoxLitO'Sh92].

Define a **left Gröbner basis** for a left ideal $I \subseteq \wedge$ to be a set $G = \{g_1, \dots, g_r\} \subset I$ such that G left-generates I and $\{\text{in}_{\prec}(g_1), \dots, \text{in}_{\prec}(g_r)\}$ left-generates $\text{in}_{\prec}(I)$. This condition is straightforwardly equivalent to requiring that $f \in \wedge$ reduces to 0 with respect to G and \prec whenever $f \in I$. By [Sto90, Theorem 6.5], this is equivalent to requiring that for any $g_i, g_j \in G$, all T_{g_i, \mathbf{x}^c} and all $\mathbf{x}^e S_{g_i, g_j}$ ($\mathbf{x}^e \in M_\wedge$) reduce to zero with respect to G and \prec . Using this condition [Sto90, Theorem 6.6] gives an algorithm, similar to Buchberger's algorithm, for extending a given generating set of I to a left Gröbner basis of I ([Sto90] calls this a *Gröbner left ideal basis*). The ideals we consider are homogeneous, so they are two-sided and we may drop the term "left".

Proof of Theorem 1.1. If \prec_c is a coherent term order, it must satisfy C_N , so $B_i \prec_c A_i$ for some $1 \leq i \leq N$. This implies $\mathbf{x}^{a_i} \in \text{in}_{\prec_c}(I)$. It therefore suffices to show that no element of a generating set of $\text{in}_{\prec_c}(I)$ divides any \mathbf{x}^{a_j} , $1 \leq j \leq N$. To this end, we use the algorithm of [Sto90, Theorem 6.6] to extend $H = \{\mathbf{x}^{b_i} - \mathbf{x}^{a_i} : 1 \leq i \leq N\}$ to a Gröbner basis $G = \{g_1, \dots, g_r\}$ of I with respect to \prec . We will show the elements of $G \setminus H$ involve monomials only of the form $\pm \mathbf{x}^c \mathbf{x}^{a_i}$ where $1 \neq \mathbf{x}^c \in M_\wedge$. This will suffice since $\text{in}_{\prec}(I) = \langle \text{in}_{\prec}(g_j) : g_j \in G \rangle$ and by the antichain condition, none of \mathbf{x}^{b_i} , $\mathbf{x}^c \mathbf{x}^{a_i}$ (where $\mathbf{x}^c \neq 1$) divide any \mathbf{x}^{a_j} , $1 \leq j \leq N$.

Let $g \in H$ and $\mathbf{x}^c \in M_\wedge$ with $\mathbf{x}^c \text{in}_{\prec}(g) = 0$. Then $\mathbf{x}^c \neq 1$ and $T_{g, \mathbf{x}^c} = \mathbf{x}^c \mathbf{x}^{a_i}$ (possibly equal to zero). So the remainder on reducing T_{g, \mathbf{x}^c} with respect to H and remainders of other T -polynomials is either zero or also a monomial of this form. Any T -polynomial of a remainder of a T -polynomial is zero.

Let H' be the union of H and the set of nonzero remainders of T -polynomials. The S -polynomial of two elements of $H' \setminus H$ is zero. Let $g_1 \in H, g_2 \in H' \setminus H, \mathbf{x}^e \in M_\Lambda$. By the antichain condition g_2 does not divide $\text{in}_<(g_1)$, so $\mathbf{x}^e S_{g_1, g_2}$ is either zero or a monomial of the form $\pm \mathbf{x}^c \mathbf{x}^{a_i}$ for some $1 \neq \mathbf{x}^c \in M_\Lambda$. This reduces similarly to the T -polynomials.

Let $g_1, g_2 \in H, \mathbf{x}^e \in M_\Lambda$. Then S_{g_1, g_2} has the form $(-1)^{d_1} \mathbf{x}^{c_1} \mathbf{x}^{a_j} - (-1)^{d_2} \mathbf{x}^{c_2} \mathbf{x}^{a_i}$ (one or both of these terms may be zero), where $\mathbf{x}^{c_1}, \mathbf{x}^{c_2} \in M_\Lambda$. By the antichain condition $\mathbf{x}^{b_i} \nmid \mathbf{x}^{b_j}$ and $\mathbf{x}^{b_j} \nmid \mathbf{x}^{b_i}$, so $\mathbf{x}^{c_1} \neq 1$ and $\mathbf{x}^{c_2} \neq 1$. Thus reducing $\mathbf{x}^e S_{g_1, g_2}$ with respect to H' and nonzero remainders of other S -polynomials yields either 0, or a monomial or binomial with term(s) of the form $\pm \mathbf{x}^c \mathbf{x}^{a_i}$, where $1 \neq \mathbf{x}^c \in M_\Lambda$. The S -polynomial of any two such nonzero remainders, or such a nonzero remainder and an element of H' , is either zero or a monomial or binomial with terms of the form $\pm \mathbf{x}^c \mathbf{x}^{a_i}$ for some $1 \neq \mathbf{x}^c \in M_\Lambda$, and the product of \mathbf{x}^e and this S -polynomial reduces similarly. \square

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